# Two Observers:

Setup: There are two observers embeded at the same point in a plane $X$. $O_2$'s coordinate system is $O_1$'s rotated by $\theta$.

The measurement functions for the two observers are $m_1, m_2 : X \xrightarrow{\cong} \mathbb{R}^2$

& Denote $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v$

T:

$$\begin{tikzcd} & X \arrow[dl, "m_2"'] \arrow[dr, "m_1"] & \\ \mathbb{R}^2 \arrow[rr, "R_\theta"] \arrow[dr, "\|\cdot\|"'] & & \mathbb{R}^2 \arrow[dl, "\|\cdot\|"] \\ & \mathbb{R} & \end{tikzcd}$$

Euclidean norm $\mathbb{R}^2$

$m_1$ & $m_2$ give the coordinates of an element of $X$ in $O_i$ coordinate system.

This diagram commutes.

D: $f : \mathbb{R}^2 \to \mathbb{R}$ is coordinate independent if for any observers $O_1$ & $O_2$
$f(m_1(x)) = f(m_2(x))$ $\forall x \in X$.

- $\|\cdot\|$ is coordinate independent.

T: For a function $f : \mathbb{R}^2 \to \mathbb{R}$
$f$ is coordinate independent
$\iff f \circ R_\theta = f \quad \forall \theta \in \mathbb{R}$
$\iff \exists g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f = g \circ \|\cdot\|$

So $f$ is coordinate independent iff it is a function of the euclidean distance.

For pairs of points $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is coordinate independent iff
$\exists h : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times [0, 2\pi) \longrightarrow \mathbb{R}$ such that
$f(v, w) = h(\|v\|, \|w\|, \underline{\measuredangle(v,w)})$
↳ Angle between the two.

# Metric Spaces:

D: A metric space is a pair $(X, d)$ consisting of a set $X$ & a function $d : X \times X \to \mathbb{R}$ st.

(M1) $d(x,y) \geq 0$ — Non negative.

(M2) $d(x,y) = 0 \iff x = y$ — seperation

(M3) $d(x,y) = d(y,x)$ — Symmetry

(M4) $d(x,y) + d(y,z) \geq d(x,z)$ — Trinange

$\forall x, y, z \in X$.

$\emptyset$ is a metric space

$\{*\}$ is a metric space, $d(*,*) = 0$

T: The following are metrics on $\mathbb{R}^n$

- $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$
- $d_2(x,y) = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$ — Euclidean
- $d_\infty(x,y) = \max \{ |x_i - y_i| \}_{i \in [n]}$

T: $(S_1, d_a)$ is a metric space

$S_1 = \{ z \in \mathbb{R}^2 \mid \|z\| = 1 \}$

$d_a : S_1 \to [0, 2\pi)$,
$(x,y) \longmapsto \min \{ |\phi^{-1}(x) - \phi^{-1}(y)|, 2\pi - |\phi^{-1}(x) - \phi^{-1}(y)| \}$
with $\phi(x) = (\cos(x), \sin(x))$.

D: A function $f : X \to Y$ between two metric spaces $(X, d_X)$ & $(Y, d_Y)$ is distance preserving if
$\forall x_1, x_2 \in X \quad d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$

D: A bijective distance preserving function is an isometry.

T: All distance preserving functions are injective.

# Topological Spaces:

**D:** A **topological space** is a pair $(X, \mathcal{T})$ where $X$ is a set & $\mathcal{T}$ is a set of subsets of $X$ st.

(T1) $\emptyset, X \in \mathcal{T}$

(T2) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

(T3) $\{V_i\}_{i \in I}$, $V_i \in \mathcal{T}\ \forall i \in I$
$\Rightarrow \bigcup_{i \in I} V_i \in \mathcal{T}$.

The topology is a set of subsets closed under FINITE intersections & ARBITRARY unions.

**D:** Such a set $\mathcal{T}$ is a **topology**

**D:** $V \in \mathcal{T} \iff V$ is **open** in the topology

**D:** $C \subseteq X$ **closed** in the topology $\iff \exists U \in \mathcal{T}$ with $C = X \setminus U$.

**T:** $(X, \mathcal{T})$ a topological space, $Y \subseteq X$
$\Rightarrow Y$ is a topological space with the induced topology
$\mathcal{T}|_Y = \{U \cap Y \mid U \in \mathcal{T}\}$

**D:** For two topologies $\mathcal{T}_1, \mathcal{T}_2$ on $X$, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

**D:** The **discrete topology** on $X$ is $\mathcal{P}(X)$.

**D:** The **indescrete topology** is $\{\emptyset, X\}$.

**T:** The discrete topology is finer than any topology, & any topology is finer than the indescrete topology.

**D:** $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ topological spaces.
A **continuous map** between the topologies is a function $f: X \to Y$ such that
$(\forall V \subseteq Y)(V \in \mathcal{T}_Y \Rightarrow f^{-1}(V) \in \mathcal{T}_X)$.

Where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is the preimage of a set.

**T:** For any topological space $(X, \mathcal{T})$
$\exists$ a bijection $cts(X, \Sigma) \to \mathcal{T}$,
$f \mapsto f^{-1}(\{1\})$
Where $\Sigma$ is the Sierpinski space.

## TOPOLOGIES ON METRIC SPACES:

Let $(X, d)$ be a metric space.

**D:** $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$
The **ball of radius $\varepsilon$** in $X$.

**T:** $\mathcal{T}_d = \{U \subseteq X \mid (\forall x \in U)(\exists \varepsilon > 0)(B_\varepsilon(x) \subseteq U)\}$
$(X, \mathcal{T}_d)$ is a topological space.

**D:** A topological space $(X, \mathcal{T})$ is **metrisable** if there exists a metric $d$ on $X$ with $\mathcal{T}_d = \mathcal{T}$.

**T:** $(X, d)$ is a metric space with associated topology $\mathcal{T}_d$
$\Rightarrow$
- $(\forall x \in X)(\forall \varepsilon > 0)(B_\varepsilon(x) \in \mathcal{T}_d)$
- Every $U \in \mathcal{T}_d$ is a union of a set of such open balls.

Open balls are open. They also form a basis.

**T:** $(X, d_X), (Y, d_Y)$ metric spaces, $\mathcal{T}_X, \mathcal{T}_Y$ associated topologies. $f: X \to Y$ is continuous
$\iff (\forall x_1 \in X)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_2 \in X)$
$(d_X(x_1, x_2) < \delta \Rightarrow d_Y(fx_1, fx_2) < \varepsilon)$

Normal metric definition agrees with topological definition of continuity.

**D:** Two metrics $d_1, d_2$ on $X$ are **Lipschitz equivilant**
$\iff (\exists h, k > 0)(\forall x, y \in X)$
$(h d_2(x,y) \leq d_1(x,y) \leq k d_2(x,y))$

**T:** This forms an equivilence relation on metrics.

**T:** If two metrics are Lipschitz equivilant then the induced topologies are the same ie. $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

**D:** Continuous map $f: X \to Y$ is a **homeomorphism** if there is a continuous map $g: Y \to X$ with $f \circ g = id_Y$, $g \circ f = id_X$.

**T:** Continuous $f$ is homeomorphism
$\iff f$ is bijection & $(\forall U \in \mathcal{T}_X)(f(U) \in \mathcal{T}_Y)$

## TOPOLOGICAL BASIS:

D: $(X, \mathcal{T})$ a topological space. A set $\mathcal{B} \subseteq \mathcal{T}$ is a **basis** for $\mathcal{T}$
$\iff (\forall U \in \mathcal{T})(\forall x \in U)(\exists B \in \mathcal{B})(x \in B \subseteq U)$

T: $\iff (\forall U \in \mathcal{T})(\exists (B_i)_{i\in I}, B_i \in \mathcal{B})(U = \bigcup_{i\in I} B_i)$

Every set in the topology can be written as a (potentially infinite/empty) union over elements of the basis.

T: $\mathcal{B}$ a basis for $\mathcal{T}_X$, $f: Y \to X$ a function for $Y$ a topological space
$\Rightarrow [f \text{ is continuous} \iff (\forall B \in \mathcal{B})(f^{-1}(B) \in \mathcal{T}_Y)]$

When you have a basis it suffices to check the preimages of your basis elements to show continuity (No longer the whole topology).

T: $X$ a set, $\mathcal{B}$ a collection of subsets of $X$ with
- $(\forall x \in X)(\exists B \in \mathcal{B})(x \in B)$
- $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2$
  $\Rightarrow (\exists B_3 \in \mathcal{B})(x \in B_3 \subseteq B_1 \cap B_2)$

THEN ($\Rightarrow$) There is a unique topology $\mathcal{T}$ on $X$ for which $\mathcal{B}$ is a basis.

D: $\mathcal{T}$ is the **topology generated** by $\mathcal{B}$.

## CREATING TOPOLOGIES:

D: $\{X_i\}_{i\in I}$ an indexed family of topological spaces. The **product space** $\prod_{i\in I} X_i$ is the product set with the topology generated by the following basis

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \;\middle|\; U_i \subseteq X_i \text{ open } \forall i \in I \text{ and } \{i \in I \mid U_i \neq X_i\} \text{ is finite} \right\}$$

The basis is ALL such products over all possible $U_i$ & all possible finite collections of them.

T: $E_i \subseteq X \times X$ an equivalence relation
$\Rightarrow \bigcap_{i\in I} E_i$ is an equivalence relation

T: $Q \subseteq X \times X \Rightarrow E = \bigcap \{Y \subseteq X \times X \mid Q \subseteq Y, Y \text{ equivalence relation}\}$ is an equivalence relation.

D: This $E$ is the **equivalence relation generated** by $Q$.

T: (Universal Property of $\prod$): $\{X_i\}_{i\in I}$ a family of topological spaces, $Y$ another topological space.
$\Rightarrow \exists$ a bijection

$$\mathrm{Cts}\left(Y, \prod_{i\in I} X_i\right) \xrightarrow[\varphi]{\cong} \prod_{i\in I} \mathrm{Cts}(Y, X_i)$$

$$\varphi(f) = (\pi_i \circ f)_{i\in I}$$

Where $\pi_j : \prod_{i\in I} X_i \to X_j$, $(x_i)_{i\in I} \longmapsto x_j$.

So given $f_i : Y \to X_i$ continuous $\exists$ a unique cnt map $f : Y \to \prod X_i$ such that $\pi_i \circ f = f_i$ for all $i$.

D: $\{X_i\}_{i\in I}$ topological spaces. The **disjoint union** or **coproduct space** $\coprod_{i\in I} X_i$ is the disjoint union set $\coprod_{i\in I} X_i = \bigcup_{i\in I} \{i\} \times X_i$ with the topology $\mathcal{T} = \{\coprod_{i\in I} U_i \mid U_i \in X_i \text{ open } \forall i \in I\}$.

D: $\iota_j : X_j \to \coprod_{i\in I} X_i$, $x \longmapsto (j, x)$
(This map is continuous).

T: (Universal Property of $\coprod$): For any space $Y$ there is a bijection

$$\mathrm{cts}\left(\coprod_{i\in I} X_i, Y\right) \longrightarrow \prod_{i\in I} \mathrm{cts}(X_i, Y)$$

D: $X$ a topological space. $\sim$ An equivalence relation on $X$. The **quotient space** $X/\sim$ is

$$X/\sim \; = \{[x] \mid x \in X\}$$

Where $[x] = \{y \in X \mid x \sim y\}$

With the topology given by the quotient map $\rho : X \to X/\sim$, $x \longmapsto [x]$, ie.
$\mathcal{T} = \{U \subseteq X/\sim \mid \rho^{-1}(U) \text{ open in } X\}$

T: For any space $Y$ & continuous $f : X \to Y$ such that $[f(x_1) = f(x_2) \iff x_1 \sim x_2]$
$\Rightarrow \exists$ unique continuous map $F$ with commuting

$$\begin{tikzcd}
X \arrow[r, "\rho"] \arrow[dr, "f"'] & X/\sim \arrow[d, "F"] \\
& Y
\end{tikzcd}$$

**D:** For a pair of cnt maps $f: X \to Y$, $g: X \to Z$ we define the **pushout** of $f, g$ as the space

$$Y \amalg_X Z = (Y \amalg Z)/\sim$$

with $\sim$ the smallest equivilence relation such that $\forall x \in X$ $(fx, gx) \in \sim$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\nu_1"] & \\ Z \arrow[r, "\nu_2"'] & Y \amalg Z \arrow[dr, "\rho"] & \\ & & Y \amalg_X Z \end{tikzcd}$$

**T:** (Universal Property of Pushout):
For given $f$ & $g$ as above and cnt maps $u: Y \to W$ and $v: Z \to W$ such that this diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] & W \end{tikzcd}$$

ie. $u \circ f = v \circ g$

Then there is a unique cnt map $t: Y \amalg_X Z \to W$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\nu_Y"] \arrow[ddr, bend left, "u"] & \\ Z \arrow[r, "\nu_Z"] \arrow[drr, bend right, "v"'] & Y \amalg_X Z \arrow[dr, "t"] & \\ & & W \end{tikzcd}$$

ie. Such that the two markes subtriangles of the diagram commute.

**D:** For $n \geq 0$

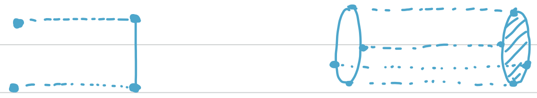
$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ "n sphere"

$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ "n Disk"

Note that $\|\cdot\|$ is euclidean $d_2$ norm, & $D^0 = \{*\}$.

**D:** For $n \geq 1$ we denote the **inclusion** $\iota: S^{n-1} \to D^n$.

$S^0 \xhookrightarrow{\iota} D^1$ $\qquad$ $S^1 \hookrightarrow D^2$



**D:** A topological space $Y$ is obtained from topological space $X$ by **attaching n-cells** $(n \geq 1)$ if $\exists$ a family of continuous maps $\{f_\alpha : S^{n-1} \to X\}_{\alpha \in \Lambda}$ and a homeomorphism between $X$ & the pushout of ...

$f = (f_\alpha)_{\alpha \in \Lambda}$

$$\begin{tikzcd} \coprod_{\alpha \in \Lambda} S^{n-1} \arrow[r, "f"] \arrow[d, "\coprod_\alpha \iota"'] & X \arrow[d] & \\ \coprod_{\alpha \in \Lambda} D^n \arrow[r] & X \amalg_{\coprod_\alpha S^{n-1}} \coprod_{\alpha \in \Lambda} D^n \arrow[r, phantom, "\cong"] & Y \end{tikzcd}$$

homeomorphic

Pushout

Where $f|_{S^{n-1}} = f_\alpha$, $\iota: S^{n-1} \to D^n$ is the inclusion.

$Y$ is obtained from $X$ by "gluing in" n cells along attaching maps $f_\alpha$.

Note that $\Lambda$ may be empty.

**D:** A topological space $X$ is a **finite CW complex** if $\exists$ a sequence $X_0, \dots, X_n = X$ of topological spaces where $X_0$ is a finite set with discrete topology & $X_i$ is obtained from $X_{i-1}$ by attaching a finite # of $i$-cells.

**D:** A **presentation** of $X$ is such a sequence along with the attaching maps $\{f_\alpha : S^{i-1} \to X_{i-1}\}_{\alpha \in \Lambda_i}$ used at each step.

# Compactness:

## SEQUENTIALLY COMPACT METRIC SPACES:

D: $X \subseteq \mathbb{R}$ is bounded $\Leftrightarrow \exists M>0, X \subseteq [-M,M]$

D: For $X \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is an adherent point of $X$

$\Leftrightarrow$ $\exists$ a sequence $(a_n)_{n=0}^{\infty}$ converging to $x$ with $a_i \in X$ $\forall i$

$\Leftrightarrow \forall \varepsilon>0\ \exists y \in X\ (|x-y|<\varepsilon)$

T: $X$ is closed if it contains all its adherent points. This holds in any metric space (closed in the metric topology).

T: (Bolzano Weierstrass) $K \subseteq \mathbb{R}$ is closed and bounded iff every sequence in $K$ contains a convergent subsequence, converging to a point in $K$.

D: $(X,d)$ a metric space, $(x_n)_{n=0}^{\infty}$ a sequence in $X$. $(x_n)_{n=0}^{\infty}$ converges to $x \in X$

$\Leftrightarrow \lim_{n\to\infty} x_n = x$

$\Leftrightarrow \forall \varepsilon>0\ \exists N>0\ \forall n \in \mathbb{N}\ (n \geq N \Rightarrow d(x_n,x)<\varepsilon)$

$\Leftrightarrow \forall \varepsilon>0\ \exists N>0\ (\{a_n\}_{n\geq N} \subseteq B_\varepsilon(x))$

T: If $(x_n)_{n=0}^{\infty}$ has a limit it is unique.

T: A function between two metric spaces $f:(X,d_X) \longrightarrow (Y,d_Y)$ is cts

$\Leftrightarrow [x_n \longrightarrow x \text{ in } X \Rightarrow f(x_n) \longrightarrow f(x) \text{ in } Y]$

D: $(X,d)$ a metric space is sequentially compact if every sequence in $X$ has a convergent subsequence.

D: A subset $K \subseteq X$ is sequentially compact if the metric space $(K, d|_{K\times K})$ is sequentially compact.

T: $f:(X,d_X) \longrightarrow (Y,d_Y)$ is cts $K \subseteq X$ sequentially compact $\Rightarrow f(K) \subseteq Y$ sequentially compact.

T: $f: X \longrightarrow \mathbb{R}$ on nonempty sequentially compact metric space $(X,d)$

$\Rightarrow \exists \lambda_1, \lambda_2 \in X$ with

$\forall x \in X \quad f(\lambda_1) \geq f(x) \geq f(\lambda_2)$

Extreme value theorem.

D: $Y \subseteq X$ a metric space $(X,d)$ is bounded if $\exists x \in X\ \exists \varepsilon>0\ \ Y \subseteq B_\varepsilon(x)$.

T: $K \subseteq X$ (a metric space) is sequentially compact $\Rightarrow$ $K$ is closed & bounded in $X$.

T: $(X,d)$ sequentially compact, $Y \subseteq X$ closed $\Rightarrow$ $Y$ is sequentially compact

## COMPACT TOPOLOGICAL SPACES:

D: $(X,\mathcal{T})$ a topological space. $\mathcal{C} = \{U_i\}_{i\in I}$ an indexed family of open sets. $\mathcal{C}$ covers $X$ (or $\mathcal{C}$ forms an open cover) if $X = \bigcup_{i\in I} U_i$

D: $\mathcal{C}$ covers $Y \subseteq X$ if $\{U_i \cap Y\}_{i\in I}$ covers $Y$.

D: $\mathcal{C}$ is finite if $I$ is finite

D: A subcover of $\mathcal{C}$ is an indexed set $\{U_j\}_{j\in J}$ with $J \subseteq I$, which is itself a cover.

D: A topological space $X$ is compact if every cover of $X$ has a finite subcover.

T: $\beta$ a basis for topological space $X$. $X$ is compact $\Leftrightarrow$ every open cover consisting of sets in $\beta$ has a finite subcover.

T: $(X,d)$ sequentially compact metric space. $\forall \varepsilon>0\ \ \exists x_1,\dots,x_n \in X$ such that $\{B_\varepsilon(x_i)\}_{i=1}^{n}$ covers $X$.

T: $(X,d)$ metric space with associated topological space $(X,\mathcal{T})$. $(X,d)$ is sequentially compact $\Leftrightarrow$ $(X,\mathcal{T})$ is compact.

T: $K \subseteq X$ a topological space is compact $\Leftrightarrow$ For every indexed family of open sets $\{U_i\}_{i\in I}$ such that $K \subseteq \bigcup_{i\in I} U_i$ $\exists$ a finite $I' \subseteq I$ with $K \subseteq \bigcup_{i\in I'} U_i$

T: $f: X \longrightarrow Y$ cts $K \subseteq X$ compact $\Rightarrow f(K) \subseteq Y$ is compact.

T: $f: X \longrightarrow \mathbb{R}$ cts on nonempty compact topological space $X$.
$\Rightarrow \exists c, d \in X$ with $f(c) \geq f(x) \geq f(d)$ $\forall x \in X$.

T: Every closed subspace of a compact topological space is compact.

T: For $X$ & $Y$ compact topological spaces
- $X/\sim$ compact
- $X \times Y$ compact
- $X \amalg Y$ compact

T: Any finite CW complex is compact

T: (Heine-Borel):
$X \subseteq \mathbb{R}^n$ compact $\Longleftrightarrow$ $X$ is closed & bounded

T: $D^n \subseteq \mathbb{R}^n$ & $S^n \subseteq \mathbb{R}^{n+1}$ are compact
T: For $Y_1, \dots, Y_n \subseteq X$ some space such that $\forall i$ $Y_i$ is compact
$\Longrightarrow \bigcup_{i=1}^{n} Y_i$ is compact subset of $X$.

D: Topological space $X$ is locally compact
$\Longleftrightarrow \forall x \in X$ $\exists U \subseteq X$ open $\exists K \subseteq X$ compact such that $x \in U \subseteq K$.

T: $X$ locally compact
- $\Rightarrow [A \subseteq X$ closed $\Rightarrow A$ locally compact$]$
- $\Rightarrow [X$ Hausdorff $\Rightarrow X$ regular$]$

## HAUSDORFF SPACES & SEPERATION CONDITIONS

D: Topological space $X$ is Hausdorff if for any $x, y \in X$ $x \neq y$ $\exists U, V$ open with $x \in U$ & $y \in V$ and $U \cap V = \emptyset$.

T: $X$ Hausdorff, $x \in X$
$\Longrightarrow \{x\}$ closed

T: $X$ metrisable $\Longrightarrow X$ Hausdorff
T: $\{X_i\}_{i \in I}$ a family of Hausdorff spaces $\Longrightarrow \prod_{i \in I} X_i$ is Hausdorff
T: $X$ & $Y$ Hausdorff $\Rightarrow X \amalg Y$ Hausdorff
T: Any compact subspace of a Hausdorff space is closed.
T: $X$ compact, $Y$ Hausdorff. Then any continuous bijection $f: X \longrightarrow Y$ is a homeomorphism

D: $f: X \longrightarrow Y$ is open when
$U \subseteq X$ open $\Longrightarrow f(U)$ open
maps open sets to open sets

T: Any finite CW complex is compact Hausdorff.

D: Suppossing one point sets are closed in $X$
- $X$ is regular if for each point $x$ and closed $B \subseteq X$ with $x \notin B$ there exists disjoint open sets $U, V$ such that $x \in U$, $B \subseteq V$.
- $X$ is normal if for each closed disjoint pair of sets $A, B \subseteq X$ $\exists U, V$ open and disjoint $A \subseteq U$, $B \subseteq V$.

T: Any metrisable space is normal.
T: Any compact Hausdorff space is normal

# Function Spaces:

## SUB-BASIS

**D:** The topology on $X$ generated by a collection of subsets $S \subseteq \mathcal{P}(X)$ is
$\langle S \rangle = \bigcap \{ \mathcal{T} \mid \mathcal{T}$ is a topology on $X$ & $S \subseteq \mathcal{T} \}$.

**D:** $(X, \mathcal{T})$ topological space. A sub basis of $\mathcal{T}$ is any $\mathcal{Q} \subseteq \mathcal{T}$ such that $\langle \mathcal{Q} \rangle = \mathcal{T}$.

**T:** $U \in \langle S \rangle \iff U = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} S_{j,i}$, $S_{j,i} \in S \ \forall j, i$

Any set in the topology is expressable as a union of finite intersections of elements of the generating set.

**T:** $f : X \longrightarrow Y$. $S$ a subbasis for the topology on $Y \Rightarrow \left[ f \text{ is cts} \iff f^{-1}(U) \text{ open } \forall U \in S \right]$

## THE COMPACT OPEN TOPOLOGY:

**D:** $X$ & $Y$ topological spaces. The compact open topology $\mathcal{T}_{X,Y}$ on $\mathrm{cts}(X,Y)$ is the topology generated by the set $\{S(K,U)\}_{K \subseteq X \text{ compact}, U \subseteq Y \text{ open}}$. where $S(K,U) = \{ f \mid f(K) \subseteq U \}$.

ie. $\mathcal{T}_{X,Y} = \langle \{ S(K,U) \}_{K \subseteq X \text{ compact}, U \subseteq Y \text{ open}} \rangle$

**T:** With $\mathcal{T}_{X,Y}$ we have that for any cts $F : Z \times X \longrightarrow Y$, the map $z \longmapsto F(z, -)$ is a cts map $Z \longrightarrow \mathrm{cts}(X,Y)$ AND for $X$ locally compact Hausdorff there is a bijection

$$\mathrm{cts}(Z \times X, Y) \xrightarrow{\Psi_{Z,X,Y}} \mathrm{cts}(Z, \mathrm{cts}(X,Y))$$
$$\Psi_{Z,X,Y}(F)(z)(x) = F(z,x)$$

The existence of this bijection is the adjunction property.

**T:** $C_{X,Y,Z} : \mathrm{cts}(Y,Z) \times \mathrm{cts}(X,Y) \longrightarrow \mathrm{cts}(X,Z)$
$(g,f) \longmapsto g \circ f$
is cts whenever $X$ & $Y$ are locally compact Hausdorff.

**T:** $f : X \to Y$ cts, $Y$ locally compact Hausdorff $\Rightarrow \mathrm{cts}(Y,Z) \longrightarrow \mathrm{cts}(X,Z)$
$g \longmapsto g \circ f$
is cts for any $Z$.

**T:** $g : Y \to Z$ cts, $X$ locally compact Hausdorff
$\Rightarrow \mathrm{cts}(X,Y) \longrightarrow \mathrm{cts}(X,Z)$ is cts.
$f \longmapsto g \circ f$

(compositions continuous under certain conditions).

**T:** For $X$ compact Hausdorff $\{\chi_X\}$ is open in $\mathcal{T}_{X,\Sigma}$: where $\chi_X$ is the characteristic function of $X$ & $\mathcal{T}_{X,\Sigma}$ is the compact open topology on $\mathrm{cts}(X, \Sigma)$

Sierpinski Space.

**T:** $Y$ Hausdorff $\Rightarrow \mathrm{cts}(X,Y)$ Hausdorff

## CLOSURE:

**D:** $A \subseteq X$ for $X$ a topological space

- $\bar{A} = \bigcap \{ C \subseteq X \mid C \text{ is closed } \& \ A \subseteq C \}$
- $A^\circ = \bigcup \{ U \subseteq X \mid U \text{ is open } \& \ U \subseteq A \}$

$\bar{A}$ is the closure of $A$. $A^\circ$ is the interior of $A$.

**T:** Some properties of closure

- $x \in \bar{A} \iff$ Every open neighbourhood of $x$ contains an element of $A$.
- For a metric space $(X,d)$, $\overline{B_\varepsilon(x)} \subseteq \{ y \in X \mid d(x,y) \leq \varepsilon \}$
- $f : X \to Y$ cts $\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$
- $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

## METRICS ON FUNCTION SPACES:

T: $X$ compact, $Y$ metrisable. THEN for any metric $d_Y$ inducing the topology on $Y$, there is an associated metric on $Cts(X,Y)$

$$d_\infty(f,g) = \sup\{d_Y(fx, gx) \mid x \in X\}$$

which gives the compact open topology.

Note that the topology on $Cts(X,Y)$ is independent of the choice of metric on $Y$.

## COMPLETENESS & FIXED POINTS:

D: $(Y,d)$ metric space. $A \subseteq Y$. For $y \in Y$ we define $d(y,A) = \inf\{d(y,a) \mid a \in A\}$

T: $d(-,A): Y \longrightarrow \mathbb{R}$ is cts.

T: $(Y, d_Y)$ a metric space, $K$ compact, $U$ open st. $K \subseteq U \Rightarrow \exists \varepsilon > 0\ \forall k \in K\ \forall x \notin U\ d_Y(x,k) > \varepsilon$

D: $X$ a set $(Y, d_Y)$ a metric space $(f_n)_{n\geq 0}$ a sequence of functions with $f_n : X \longrightarrow Y$. Let $f: X \longrightarrow Y$ a function

- $(f_n)_{n\geq 0}$ converges pointwise to $f$
$\Leftrightarrow \forall x \in X\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ (n \geq N \Rightarrow d_Y(f_n x, fx) < \varepsilon)$

- $(f_n)_{n\geq 0}$ converges uniformly to $f$
$\Leftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in X\ (n \geq N \Rightarrow d_Y(f_n x, fx) < \varepsilon)$

T: $X$ a topological space, $(Y, d_Y)$ metric space $f: X \longrightarrow Y$ is the uniform limit of $(f_n)_{n\geq 0}$
THEN: $f_n$ cts $\forall n \Rightarrow f$ cts.

D: A metric space $(A,d)$ is complete if every cauchy sequence in $A$ converges to a value in $A$.

Note that completeness is a genuine property of the metric Not the topology.

T: If two metrics on $A$, $d_1$ & $d_2$, are Lipschitz equivilent then
$(A, d_1)$ complete $\Leftrightarrow$ $(A, d_2)$ complete

T: $(A,d)$ complete, $B \subseteq A$ closed
$\Rightarrow (B,d)$ complete.

T: Any compact metric space is complete.

T: $X$ compact, $(Y, d_Y)$ complete
$\Rightarrow (\underbrace{Cts(X,Y)}_{\text{Topological space}}, \underbrace{d_\infty}_{\text{metric spaces}})$ complete

D: A fixed point of a function $f: X \longrightarrow X$ is an $x \in X$ with $fx = x$

D: $(X,d)$ metric space. $f: X \longrightarrow X$ is a contraction mapping if $\exists \lambda \in (0,1)$
$d(fx, fx') \leq \lambda d(x, x')\ \forall x, x' \in X$.

D: $\lambda$ is the contraction factor

T: Any contraction mapping is cts.

T: (Banach Fixed point Thm):
$(X,d)$ complete. $f: X \to X$ contraction map
$\Rightarrow f$ has a unique fixed point.
AND $\forall x \in X$ $(f^n x)_{n\geq 0}$ converges to this unique fixed point.

T: (Picard): $h: U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}^2$ open.
st $(x_0, y_0) \in U$. and
$(\exists \alpha > 0)(\forall (x,y_1),(x,y_2) \in U)$
$(|h(x,y_1) - h(x,y_2)| \leq \alpha |y_1 - y_2|)$

$\Longrightarrow \exists \delta > 0$ such that initial Value problem $\varphi'(x) = h(x, \varphi(x))$, $\varphi(x_0) = y_0$ has a unique solution on $[x_0 - \delta, x_0 + \delta]$.

# TUTORIALS:

## 1

D: $(x,y) = \{\{x\},\{x,y\}\}$
The "Kuratowski pair" def

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

$\prod_{i\in I} X_i = \{(x_i)_{i\in I} \mid x_i \in X_i\}$

D: $(x_i)_{i\in I} = \{(i,x_i) \mid i \in I\}$

(Cartesian product).

D: Disjoint union

$$\coprod_{i\in I} X_i = \bigcup_{i\in I} \{i\} \times X_i$$

$$= \{(i,x) \mid i \in I, x \in X_i\}$$

T: $U_i, V_i \subseteq X_i$

$$\Rightarrow \left(\coprod_i U_i\right) \cap \left(\coprod_i V_i\right) = \coprod_i (U_i \cap V_i)$$

## 2

T: $f : X/\sim \;\to Y$ is cts $\iff f \circ p$ is cts.

T: There is a bijection between open sets of $X/\sim$ & saturated open sets of $X$.

D: A saturated open set $U \subseteq X$ is a set that

- Is open
- $x \sim y$, $x \in U \Rightarrow y \in U$

## 3

D: A topological group is a set $X$ with both a topology, $\mathcal{T}$, & group properties $(X, \cdot, e)$
functions, identity
such that

$\cdot : X \times X \to X$ is cts

$(-)^{-1} : X \to X$ is cts

D: An isomorphism of Topological groups is an isomorphism of groups that is also a homeomorphism.

## 4

T: $X$ locally compact Hausdorff $\Rightarrow X$ is homeomorphic to a subspace of a compact Hausdorff space.

D: For LCH space $X$ we define the one-point-compactification $\tilde{X} = X \amalg \{\infty\}$ which is compact Hausdorff.

- $U \subseteq \tilde{X}$, $\infty \notin U$ open $\iff U$ open in $X$.
- $U \subseteq \tilde{X}$, $\infty \in U$ open $\iff \exists K \subseteq U$ compact with $U = K^c \amalg \{\infty\}$.

## 5

Paths

## 6